Motion of charged particles around a rotating black hole in a magnetic field

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February 7, 2008

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ABSTRACT

We study the effects of an external magnetic field, which is assumed to be uniform at infinity, on the marginally stable circular motion of charged particles in the equatorial plane of a rotating black hole. We show that the magnetic field has its greatest effect in enlarging the region of stability towards the event horizon of the black hole. Using the Hamilton-Jacobi formalism we obtain the basic equations governing the marginal stability of the circular orbits and their associated energies and angular momenta. As instructive examples, we review the case of the marginal stability of the circular orbits in the Kerr metric, as well as around a Schwarzschild black hole in a magnetic field. For large enough values of the magnetic field around a maximally rotating black hole we find the limiting analytical solutions to the equations governing the radii of marginal stability. We also show that the presence of a strong magnetic field provides the possibility of relativistic motions in both direct and retrograde innermost stable circular orbits around a Kerr black hole.

Key words: gravitation, accretion discs - black hole physics - magnetic fields.

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1 Introduction

New observational evidence for black holes provides new motivations for the investigation of the general relativistic dynamics of particles and electromagnetic fields in the vicinity of the black holes. We shall start with a brief description of the situation. The results of astronomical observations over the last decade continue to point insistently to the existence of stellar-mass and supermassive black holes in some X-ray binary systems and in galactic centres (see Horowitz & Teukolsky 1999; Rees 1998 for reviews). The typical examples of the stellar-mass black holes in X-ray binaries are Cyg X-1 discovered back in 1971, the X-ray source LMC X-3 in the Large Magellanic Cloud, as well as the source in A0620-00 discovered in 1975 and a number of recently discovered sources, such as V404 Cyg, GS 2000+25, GRO J0422+32 (for full list see Charles 1999). New observational data, such as the detection of broad iron fluorescence lines and maser emission lines of water in the spectra provide the strongest suggestion for the presence of supermassive black holes in the centres of the active galaxies MCG 6-30-15, NGC 4258 and NGC 1068 Menou, Quataert & Narayan 1999; Rees 1998; Miyoshi et.al 1995; Watson & Wallin 1994).

The supermassive black holes are also strongly believed to be in the centres of some low-level active, or non-active, galaxies. For instance, recent progress in the studies of the distributions and velocities of stars near the centres of the giant elliptical galaxy M87, Andromeda M31 and our own Galaxy have revealed the strong evidence for the existence of the supermassive black holes in these centres (Merritt & Oh 1997; Richstone et al. 1998; Rees 1982, 1998).

On the other hand, a convincing explanation for a huge amount of energy output from the active cores of the galaxies associated with the supermassive black holes requires the searches for mechanisms responsible for the high-level energy release. One of these mechanisms is the extraction of the rotational energy from a rotating black hole surrounded by stationary magnetic fields. The magnetic fields threading the event horizon tap the rotational energy of the black hole due to the interaction between charged particles and the induced electric field (Blandford & Znajek 1977; Thorne, Price & Macdonald 1986). The interest in this model still continues to point out new gravito-electromagnetic phenomena in the strong and weak field domains around a rotating black hole (Bičák & Ledvinka 2000; Chamblin, Emparan & Gibbons

1998; Mashhoon 2001; see also Punsly 2001). Another mechanism responsible for high-level energy release is gas accretion by a black hole (see Shapiro & Teukolsky 1983), where energy is released mostly at the expense of the binding energy of the particles and the strong gravitational field of the black hole. It is well known that the binding energy in the innermost stable orbits of the particles determines the potential efficiency of an accretion disc. It is about 5.7 % of the rest energy in the Schwarzschild field, but in the case of a maximally rotating black hole it approaches 42 % of the rest energy. The observational data from the core of some galaxies, such as the elliptical galaxy M87 reveal that the inner part of a gas disc around a putative supermassive black hole emits non-thermal radiation and radio waves. The reason for this is believed to be synchrotron emission from ultrarelativistic electrons moving in a strong magnetic field in the inner part of the accretion flow onto the supermassive black hole (Fabian & Rees 1995; Narayan & Yi 1995; Rees 1998). This gives us a new impetus to return back once again to the investigation of the motion of charged particles in the model of a rotating black hole in a uniform magnetic field, though much insight into this problem was given in 1980s (Prasanna & Vishveshwara 1978; Prasanna 1980; Wagh, Dhurandeur & Dadhich 1985; Aliev & Gal'tsov 1989; see also Frolov & Novikov 1998 and references therein). In particular, the works of Prasanna & Vishveshwara (1978) and Prasanna (1980) have given a comprehensive numerical analysis of the charged particle motion in a magnetic field superposed on the Kerr metric by studying the structure of the effective potential for radial motion and integrating the complete set of equations of the motion for appropriate initial conditions.

In this paper we shall study a particular class of orbits, namely the marginally stable circular orbits of charged particles in the equatorial plane of a Kerr black hole immersed in a uniform magnetic field. The numerical analysis performed by Prasanna & Vishveshwara (1978) has revealed that the presence of a uniform magnetic field on the Kerr background increases the range of stable circular orbits. This result is confirmed in our model of the marginally stable circular motion, however we also obtain significant new results: First of all, we derive the basic equation determining the region of the marginal stability of the circular orbits, that comprises only two parameters, namely the rotation parameter of black hole and the strength of the magnetic field. We also find the closed analytical expressions for the associated angular momentum and energy of a charged particle moving in

a marginally stable circular orbit. Further, for a sufficiently large values of the magnetic field, as well as for a maximum value of the black hole rotation parameter we find the limiting analytical solutions for the radii of stability of both direct and retrograde innermost circular orbits. This allows us not only to emphasize that the presence of a magnetic field enlarges the region of stability towards the event horizon, but also to find the limiting values for this enlargement both for direct and retrograde motions along with their associated energies and angular momenta. Our analytical and numerical calculations show that the combined effects of a sufficiently strong magnetic field and the rotation of a black hole give rise to the possibility of relativistic motion of the charged particles in the innermost stable direct and retrograde orbits. The existence of relativistic motion in the innermost stable direct orbits is especially important new feature worked out in our analysis. These orbits lie very close to the event horizon of a rotating black hole and they may provide a mechanism for synchrotron emission from relativistic charged particles with the signatures of the strong-gravity domain.

The paper is organized as follows. We shall first review the solution of the Maxwell equations describing a uniform magnetic field in the Kerr metric with a small electric charge (see Section 2). In Section 3 we consider the separation of variables in the Hamilton-Jacobi equation and derive the effective potential for the radial motion of charged particles around a Kerr black hole in a uniform magnetic field. These results are used in Section 4 to obtain the basic equations governing the region of the marginal stability of the circular orbits and their associated energies and angular momenta. To make the analysis more transparent in the general case, we shall first present the results of analytical and numerical calculations for the marginal stability of the circular orbits in the pure Kerr metric, as well as in a uniform magnetic field in the Schwarzchild metric (see Sections 4.1 and 4.2). Section 4.3 is devoted to a comprehensive analysis of the effect of a uniform magnetic field on the radii and the assigned energy and angular momentum of the marginally stable circular orbits for various values of the magnetic field and the rotation parameter of the black hole.

2 Uniform magnetic field around a rotating black hole

It is well known that an electrically neutral black hole can not have intrinsic magnetic field (Ginzburg & Ozernoy 1965). However, a magnetic field near a black hole can arise due to external factors, such as the presence of a nearby magnetars or neutron stars. Accretion of a matter cloud may also form a magnetosphere with a superstrong magnetic field of magnitude up to $B_M = \frac{1}{M} \simeq 2 \times 10^{10}(M/10^9 M_\odot)^{-1}G$ around a supermassive black hole (Kardashev 1995). This value corresponds to the case when the magnetic and gravitational pressures become equal to each other (Bičák & Janiš 1985). It is clear that with this value of the magnetic field the space-time geometry near a black hole will be significantly distorted (Aliev & Gal'tsov 1989), however, when $B \ll B_M$ there is definitly a region near the black hole where the space-time is not distorted by the external magnetic field and the latter can be considered as a perturbation. We shall consider this case i.e. a rotating black hole with a small electric charge ($Q \ll M$) immersed in an external magnetic field described by a corresponding solution of the Maxwell equations against the background of the Kerr metric

$$
\begin{aligned}
ds^2 &= \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar}{\Sigma} \sin^2\theta \, dt \, d\phi - \frac{A\sin^2\theta}{\Sigma} \, d\phi^2 \\
&\quad -\Sigma \left(\frac{dr^2}{\Delta} + d\theta^2\right),
\end{aligned}
\tag{1}
$$

where M is the mass of the black hole, $a = J/M$ is its angular momentum per unit mass, $A = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta$, $\Delta = r^2 + a^2 - 2Mr$ and $\Sigma = r^2 + a^2 \cos^2\theta$. We shall assume a magnetic field around the Kerr black hole to be uniform at infinity. In this case there is an elegant way to construct the solution of the Maxwell equation (Wald 1974). We shall now give a succinct description of this solution.

The Kerr space-time is stationary and axially symmetric that implies the existence of two commuting Killing vector fields $\xi^\mu_{(t)} = (1, 0, 0, 0)$ and $\xi^\mu_{(\phi)} = (0, 0, 0, 1)$. These fields are used as a 4-vector potential A_μ describing electromagnetic fields in the Kerr metric that are superpositions of Coulomb electric and asymptotically uniform magnetic fields. Indeed, for Ricci-flat

space-times ($R_{\mu\nu} = 0$), the Maxwell equations for the 4-vector potential in the covariant Lorentz gauge $A^\mu_{;\,\mu} = 0$

$$A^\mu_{;\,\nu}{}^{;\,\nu} = 0 \tag{2}$$

and the equation

$$\xi^\mu_{;\,\nu}{}^{;\,\nu} = 0 \tag{3}$$

for a Killing vector are the same. The semicolon means covariant differentiation. We shall take the 4-vector potential in the form

$$A^\mu = \alpha\,\xi^\mu_{(t)} + \beta\,\xi^\mu_{(\phi)} \tag{4}$$

where α and β are arbitrary parameters. Next, we calculate the Maxwell 2-form using equation (4). We find

$$
\begin{aligned}
F \;=\;& \frac{2M}{\Sigma}\left(1 - \frac{2r^2}{\Sigma}\right)(\alpha - \beta a \sin^2\theta)(dt \wedge dr + a\sin^2\theta\, dr \wedge d\phi) \\[2mm]
& - \beta\left(2r\sin^2\theta\, dr \wedge d\phi + \frac{A}{\Sigma}\sin 2\theta\, d\theta \wedge d\phi\right) \\[2mm]
& + \frac{2Mar}{\Sigma^2}\sin^2\theta\left[(\alpha a - \beta(r^2 + a^2))\, dt \wedge d\theta\right. \\[2mm]
& \left. + (r^2 + a^2)(\alpha - \beta a \sin^2\theta)\, d\theta \wedge d\phi\right],
\end{aligned}
\tag{5}
$$

which in the asymptotic region $r \gg M$ reduces to

$$F = -2\beta r \sin\theta\,(\sin\theta\, dr \wedge d\phi + r\cos\theta\, d\theta \wedge d\phi). \tag{6}$$

It follows that $\beta = B/2$, where B is the strength of the uniform magnetic field, which is parallel to the rotation axis of the black hole. As for the remaining parameter α it can be specified using the surface integrals for the mass and angular momentum of the black hole (Bardeen, Carter & Hawking 1973)

$$M = \frac{1}{8\pi}\int \xi^\mu_{(t)}{}^{;\,\nu}d^2\Sigma_{\mu\nu} \qquad J = -\frac{1}{16\pi}\int \xi^\mu_{(\phi)}{}^{;\,\nu}d^2\Sigma_{\mu\nu} \tag{7}$$

along with the Gauss integral for its electric charge

$$Q = \frac{1}{4\pi}\int F^{\mu\nu}d^2\Sigma_{\mu\nu}\,. \tag{8}$$

Finally, we obtain

$$\alpha = aB - \frac{Q}{2M} \tag{9}$$

Thus, the 4-vector potential takes the form

$$A^\mu = \frac{1}{2} B \, \xi^\mu_{(\phi)} - \frac{Q - 2aMB}{2M} \, \xi^\mu_{(t)} \, . \tag{10}$$

We see that in addition to the usual magnetic and Coulomb parts of the electromagnetic field the 4-potential (10) also contains a contribution proportional to the rotation parameter of the black hole. It is clear that such a field appears due to the Faraday induction; a rotation of the Kerr metric produces an induced electric field, just as a field would be induced by a rotating loop in a magnetic field. The underlying geometry is such that the induced potential difference arises between the event horizon and infinitely remote point

$$\Delta\Phi = \Phi_H - \Phi_\infty = \frac{Q - 2aMB}{2M} \, . \tag{11}$$

Since an astrophysical object will rapidly neutralize its electric charge by a process of the selective accretion of charges from surrounding plasma, the potential difference (11) will vanish, or equivalently

$$\tilde{Q} = Q - 2aMB = 0 \tag{12}$$

and the black hole will acquire an inductive electric charge $Q = 2aMB$. On the other hand, in this very simple model we see that the Faraday induction may be a possible mechanism for the tapping of the rotational energy from a black hole. Namely, this mechanism lies in the basic structure of the model of a supermassive black hole for active galactic nuclei and quasars (Blandford & Znajek 1977; Kardashev 1995) .

3   Motion of charged particles

In a realistic model of gas accretion onto a black hole energy is released mostly at the expense of the binding energy of the test particles moving in a strong gravitational field of the black hole. This may be used to propose an alternative model for the interpretation of the observational data from the core of a number of galaxies (see Shapiro & Teukolsky 1983 and references

therein). Moreover, as we have mentioned above the observations of the core of some galaxies reveal the existence of non-thermal radiation and radio waves. The latter can be explained by synchrotron emission from ultrarelativistic electrons in a strong magnetic field in the innermost orbits around a supermassive black hole (Rees 1998). This makes it very important to investigate in detail the motion of charged particles around a rotating black hole in an external magnetic field. In the following we shall do it using (10) as a 4-vector potential of the external magnetic field. For convenience, we shall use the gauge in which $A^0 = 0$ at infinity. Then we have

$$\tilde{A}^0 = \frac{\tilde{Q}r(r^2 + a^2)}{\Delta\Sigma}, \qquad \tilde{A}^\phi = \frac{B}{2} + \frac{\tilde{Q}ra}{\Delta\Sigma}. \tag{13}$$

We shall study the motion of the test particles around a rotationg black hole with zero electric charge ($\tilde{Q} \to 0$) using the Hamilton-Jacobi equation

$$g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} + e\tilde{A}_\mu\right)\left(\frac{\partial S}{\partial x^\nu} + e\tilde{A}_\nu\right) = m^2, \tag{14}$$

where e and m are the charge and the mass of a test particle, respectively. Since t and ϕ are the Killing variables we can write the action in the form

$$S = -Et + L\phi + f(r, \theta), \tag{15}$$

where the conserved quantities E and L are the energy and the angular momentum of a test particle at infinity. Substituting it into equation (14) we come to the equation for unseparable part of the action

$$\Delta\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2 - \frac{A}{\Delta}E^2 + \frac{\Sigma - 2Mr}{\Delta\sin^2\theta}L^2 + \frac{4Mra}{\Delta}EL - eBL\Sigma$$
$$+ \frac{1}{4}e^2B^2A\sin^2\theta + m^2\Sigma = 0 \tag{16}$$

It is not possible to separate variables in this equation in general case, however one can separate it in the equatorial plane $\theta = \pi/2$. Then we obtain the equation for radial motion

$$r^3\left(\frac{dr}{ds}\right)^2 = V(E, L, r, \epsilon) \tag{17}$$

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where s is the proper time along the trajectory of a particle and

$$V = \left(r^3 + a^2 r + 2Ma^2\right) E^2 - (r - 2M) L^2 - 4MaEL$$
$$- \Delta r \left(1 + \frac{\epsilon L}{M}\right) - \frac{\Delta}{4M^2}\, \epsilon^2 \left(r^3 + a^2 r + 2Ma^2\right) \tag{18}$$

can be thought of as an effective potential of the radial motion. Here we have changed $E \to E/m$ and $L \to L/m$. The effective potential besides the energy, the angular momentum and the radius of the motion also depends on the dimensionless parameter

$$\epsilon = \frac{eBM}{m}\ , \tag{19}$$

which characterizes the relative influence of a uniform magnetic field on the motion of the charged particles. We shall call it as the influence parameter of the magnetic field. We note that even for small values of the magnetic field strength $(B/B_M \ll 1)$ the parameter ϵ for a particle with high charge-to-mass ratio (for instance, for electron $e/m \simeq 10^{21}$) may not be small.

4   Marginally stable circular orbits

We shall now describe a particular class of orbits, namely circular orbits that play an important role in understanding the essential features of the dynamics of test particles around a rotating black hole in a magnetic field. Physically, from the symmetry of the problem it is clear that the circular orbits are possible in the equatorial plane $\theta = \pi/2$ and to describe them one must set $\frac{dr}{ds}$ to be zero. This, in turn, requires vanishing of the effective potential (18)

$$V(E, L, r, \epsilon) = 0 \tag{20}$$

along with its first derivative with respect to r

$$\frac{\partial V(E, L, r, \epsilon)}{\partial r} = 0 \tag{21}$$

The simultaneous solution of these equations would determine the energy and the angular momentum of the circular motion in terms of the orbital

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radius, the hole's rotation parameter and the influence parameter of the magnetic field ϵ. However, the underlying expressions involve high order polynomial equations and their analytical solution is formidable and defies analysis. Therefore in Prasanna & Vishveshwara (1978) and Prasanna (1980) the authors have appealed to a numerical analysis of equations (20) -(21) for different values of the constants of motion, orbital radii, black hole's rotation parameter, as well as the strength parameter of the magnetic field.

Fortunately, the situation is changed if one wishes to restrict oneself to considering only the marginally stable circular orbits. After all, the stable orbits are most of interest astrophysically, as the binding energy of the marginally stable circular orbits is of an energy source for the potential efficiency of an accretion disc around a black hole. The stability of the circular motion requires the relation

$$\frac{\partial^2 V(E, L, r, \epsilon)}{\partial r^2} \leq 0 \tag{22}$$

where the case of equality corresponds to the marginally stable circular motion. It is clear that the simultaneous solutions of equations (20) -(22) will determine the region of stability, the associated energy and angular momentum of the circular orbits. It is remarkable that in this case one can obtain the close equation governing the stability region which depends only on the rotation parameter of the black hole and the influence parameter of the magnetic field. Below we shall show that for the limiting values of the parameters this equation admits the simple analytical solutions for the radii of stability.

From equations (21) and (22) we find that the angular momentum and the energy of a test particle can be given in the form

$$L = -\epsilon \left(r - \frac{a^2}{3r} \right) \pm \sqrt{\lambda} \tag{23}$$

$$E = \left[\eta \mp \frac{\epsilon}{M} \left(1 - \frac{2M}{3r} \right) \sqrt{\lambda} \right]^{\frac{1}{2}}, \tag{24}$$

where we have used the notations

$$\lambda = 2M \left(r - \frac{a^2}{3r} \right) + \frac{\epsilon^2}{4M^2} \left[r^2 \left(5r^2 - 4Mr + 4M^2 \right) + \right.$$

$$\left. \frac{2}{3} a^2 \left(5r^2 - 6Mr + 2M^2 \right) + a^4 \left(1 + \frac{4M^2}{9r^2} \right) \right] \tag{25}$$

and

$$\eta = 1 - \frac{2M}{3r} - \frac{\epsilon^2}{6}\left[4 - 5\frac{r^2}{M^2} - \frac{a^2}{M^2}\left(3 - \frac{2M}{r} + \frac{4M^2}{3r^2}\right)\right] \qquad (26)$$

These expressions depend on the radius of stability of a circular orbit which yet has to be determined. Substituting (23) and (24) into equation (20) we obtain the equation

$$\left(6Mr - r^2 + 3a^2 - \frac{4a^2M}{r}\right)\left(1 \mp \frac{\epsilon}{M}\sqrt{\lambda}\right)$$

$$+ \epsilon^2 \left[r^2\left(6 - \frac{4r}{M} + \frac{9}{2}\frac{r^2}{M^2} - \frac{r^3}{M^3}\right)\right.$$

$$\left. - a^2\left(3 + \frac{8}{3}\frac{r}{M} - \frac{5r^2}{M^2}\right) + \frac{3}{2}\frac{a^4}{M^2}\left(1 - \frac{2M}{3r} + \frac{8M^2}{9r^2}\right)\right]$$

$$\mp 6a\left[\sqrt{\lambda} \mp \epsilon\left(r - \frac{a^2}{3r}\right)\right]\left[\eta \mp \frac{\epsilon}{M}\left(1 - \frac{2}{3}\frac{M}{r}\right)\sqrt{\lambda}\right]^{\frac{1}{2}} = 0. \qquad (27)$$

The solution of this equation will determine the radii of the marginally stable circular orbits as functions of the rotation parameter a, as well as of the influence parameter of the magnetic field ϵ.

It should be noted that the circular motions will occur in direct and retrograde orbits depending on whether the particles are corotating $(L > 0)$, or counterrotating $(L < 0)$ with respect to the rotation of a black hole. In addition, the presence of a uniform magnetic field will produce the Larmor and anti-Larmor motions depending on whether the Lorentz force points toward a black hole, or it points in the opposite direction. We recall that we are considering only the case of parallel alignment of black hole's rotation axis and the direction of a uniform magnetic field. Therefore, as we shall see below, the Larmor motion $(L < 0)$ will occur in retrograde orbits, while the anti-Larmor motion $(L > 0)$ will accompany the direct orbits. Thus, in all expressions above the upper signs corresponds to the direct, or the anti-Larmor orbits and the lower signs refer to the retrograde, or the Larmor orbits. Next, before carrying out the full analysis of equations (23), (24) and (27), it is useful to proceed with the particular cases.

4.1 Kerr black hole

In the absence of a magnetic field the energy and the angular momentum of a test particle in the marginally stable circular orbits around a Kerr black hole are obtained from equations (23) and (24) for $\epsilon = 0$. We have

$$E = \sqrt{1 - \frac{2M}{3r}}, \qquad L = \pm \sqrt{2M\left(r - \frac{a^2}{3r}\right)}, \tag{28}$$

where, the radii $r = r_{ms}$ of the orbits satisfy the equation

$$6Mr^2 - r^3 - 4a^2M + 3a^2r \mp$$

$$2\sqrt{2}\, aM^{1/2}\left[(3r - 2M)\left(3r^2 - a^2\right)\right]^{\frac{1}{2}} = 0 \tag{29}$$

which is of a particular case of equation (27) for $\epsilon = 0$. Since the radii of stability are different for direct and retrograde motions, their appropriate energy and angular momentum given in equations (28) are different as well. The solution of (29) can be written in the form

$$r_{ms} = M\left\{3 + \sqrt{k_1 + k_2} \mp \left[2k_1 - k_2 - \frac{16(3 - k_1)}{\sqrt{k_1 + k_2}}\right]^{\frac{1}{2}}\right\} \tag{30}$$

where

$$k_1 = 3 + \frac{a^2}{M^2},$$

$$k_2 = \left(3 - \frac{a}{M}\right)\left(1 - \frac{a}{M}\right)^{\frac{1}{3}}\left(1 + \frac{a}{M}\right)^{\frac{2}{3}} +$$

$$\left(3 + \frac{a}{M}\right)\left(1 + \frac{a}{M}\right)^{\frac{1}{3}}\left(1 - \frac{a}{M}\right)^{\frac{2}{3}} \tag{31}$$

This expression agrees with that found by Bardeen et al. long ago (Bardeen et al. 1972). For a nonrotating black hole, $a = 0$, it gives $r_{ms} = 6M$, while in the maximally rotating case, $a = M$, one finds $r = M$ for the direct orbits and $r = 9M$ for the retrograde orbits. Taking these into account in (28) we find the limiting values for the energy of the direct and retrograde motions

$$E_{direct} = \frac{1}{\sqrt{3}}, \qquad E_{retrograde} = \frac{5}{3\sqrt{3}}. \tag{32}$$

In order to make the above limits more trasparent and also with the purpose of comparing them with the corresponding quantities in the presence of an external magnetic field Table 1 provides the full list for the values of the radii, of their assigned energies and angular momenta as functions of the rotation parameter of a black hole. It is important to note that the limiting value of the energy of a direct orbit determines the maximum binding energy that can be assigned to a last stable circular orbits. One can easily find that this quantity is of order of 42% of the rest energy and it determines the efficiency of an accretion disc around a maximally rotating black hole.

4.2   Schwarzschild black hole in a magnetic field

From equations (23) and (24) it is clear that the case $a = 0$ will give us the angular momentum and the energy of a charged particle moving in a marginally stable circular orbit around a nonrotating black hole immersed into a uniform magnetic field

$$L = -\epsilon\, r \pm \frac{r^{1/2}}{2M}\, \Lambda \tag{33}$$

and

$$E = \left[\left(1 - \frac{2M}{3r} \right) \left(1 \mp \frac{\epsilon\, r^{1/2}}{2M^2}\, \Lambda \right) + \frac{\epsilon^2}{6} \left(\frac{5r^2}{M^2} - 4 \right) \right]^{\frac{1}{2}} \tag{34}$$

where

$$\Lambda = \left[8M^3 + \epsilon^2 r \left(5r^2 - 4Mr + 4M^2 \right) \right]^{\frac{1}{2}}$$

and the stability radius $r = r_{ms}$ is determined by the equation

$$\left(6Mr - r^2 \right) \left(1 \mp \frac{\epsilon\, r^{1/2}}{2M^2}\, \Lambda \right) + \epsilon^2 r^2 \left(6 - \frac{4r}{M} + \frac{9}{2}\frac{r^2}{M^2} - \frac{r^3}{M^3} \right) = 0 \tag{35}$$

As we have mentioned above in this case there exist two different kind of motions depending on the directions of the Lorentz force with respect to a black hole. In turn, it is associated with two signs in equations (33)-(35). Following to papers by Gal'tsov & Petukhov (1978) and Aliev & Gal'tsov (1989) we shall distinguish the two kind of motions as the Larmor (the lower

signs in (33)-(35)) and the anti-Larmor (the upper signs in (33)-(35)) motions. In order to get more insight into these motions it is useful to obtain the expressions for the angular momentum and energy in the asymptotically flat region $r \gg M$. Solving equations (20) and (21) simultaneously for $r \gg M$ we find that for orbits with $L < 0$

$$L \simeq -\frac{eBr^2}{2m} \,, \qquad E \simeq \sqrt{1 + \left(\frac{eBr}{m}\right)^2} \,. \tag{36}$$

It follows that this kind of motion is nothing but an ordinary cyclotron rotation in a uniform magnetic field under the Lorentz force pointing towards the centre of the orbit of typical radius

$$r = \frac{mv}{eB\sqrt{1 - v^2}} \tag{37}$$

This is the Larmor motion, while in the opposite case of orbits with $L > 0$ we find that $E \to 1$, that is the anti-Larmor motion when the Lorentz force points outwards the centre of the motion may occur only in the presence of a black hole. Next, we need to solve equation (35) for r. It is of a high order polynomial equation and only for large values of ϵ ($\epsilon \gg 1$) one can find the limiting solutions

$$r_L = \frac{5 + \sqrt{13}}{2} M \,, \qquad r_A \to 2M. \tag{38}$$

Note that the radii of the marginally stable circular orbits are different for the Larmor (r_L) and anti-Larmor (r_A) motions, the effect of a uniform magnetic field shifts both of them towards the event horizon and even right up to the event horizon for the anti-Larmor motion when ϵ is large enough. These conclusions are in agreement with those made in Gal'tsov & Petukhov (1978) on the basis of numerical calculations. Substituting the radii (38) in equations (33)-(34) respectively, we find that for $\epsilon \gg 1$

$$E_L \to 5.56 \, \epsilon \, M \,, \qquad L \to -23.47 \, \epsilon \, M \tag{39}$$

for the Larmor motion and

$$E_A \to 0 \,, \qquad L \to 2\epsilon \, M \tag{40}$$

for the anti-Larmor motion. First of all, it follows from (39) that for $\epsilon \gg 1$ there exists a stable ultarelativistic motion of charged particles, while in the absence of a magnetic field as is well known, a stable motion in the field of a nonrotating black hole is only non-relativistic. In addition, the energy of the anti-Larmor motion in the innermost stable orbit tends to zero and the corresponding binding energy approaches $\simeq 100\%$ of the rest energy, in contrast to 42% that of in the field of a maximally rotating black hole. A more detailed analysis of how the above conclusions are made is given in Table 2. It exhibits the full list for the values of the radii and of the associated energies and angular momenta as functions of the influence parameter of the magnetic field ϵ.

4.3   General case

We now turn to the consideration of the combined effects of the black hole rotation and a uniform magnetic field on the marginally stable circular orbits. The radii of their stability are determined by equation (27), which in the general case can be solved only numerically. However, in the case of a maximally rotating black hole and large enough values of the magnetic field strength ($\epsilon \gg 1$) we find the limiting analytical solution for a retrograde motion

$$r = 2M \left\{ 1 + 2 \cos \left[\frac{1}{3} tan^{-1} \left(\frac{\sqrt{7}}{3} \right) \right] \right\} \tag{41}$$

while, for a direct motion we have the two different radii

$$r_1 \to M , \qquad r_2 = (1 + \sqrt{2}) \, M. \tag{42}$$

The appearance of the two stable orbits can apparently be related to the effect of the expelling of magnetic flux lines from the horizon of a black hole as the rotation parameter of the hole increases (Bičák & Janiš 1985). In our case a large enough value of the magnetic field shifts the innermost stable circular orbits to the event horizon and this, in turn, along with the increase of the hole's rotation parameter provides the expulsion of the magnetic field lines from the black hole. As a result of this the above two anti-Larmor orbits appear. In the case of the first orbits the radius of stability tends the event horizon, but the effect of the magnetic field on it should become less and less

as it approaches the horizon. In the second case the anti-Larmor motion is expelled from the black hole and it occurs at the limiting radius given by r_2 in (42).

A more precise description of it, of course, requires the detailed numerical analysis of equation (27). The results are listed in Tables 3.1-3.2. Comparing these results and also the radii (41)-(42) with those given in subsection 4.1 we see that the magnetic field always plays a stabilizing role in its effect on the circular orbits. In the Kerr metric the region of stability for direct orbits enlarges towards the event horizon with the increase of hole's rotation parameter, while for prograde orbits it moves in the opposite direction (see Table 1). The presence of a large-scale uniform magnetic field around a Kerr black hole shifts the innermost stable circular orbits towards the horizon for both direct and retrograde motions.

However, as it is seen from Table 3.2, when ϵ is large enough the direct motion decays into two motions and the separation becomes more significant as the rotation parameter tends to its maximum value. This is due to the effect of the expelling of the magnetic field from nearby region of the horizon as the angular momentum of the black hole increases. At the same time, Tables 3.1-3.2. show that for retrograde orbits the rotation of a black hole opposes to the stabilizing effect of the magnetic field, leading finally to the limiting value of the radius given in (41).

Some results of the numerical analysis of equation (27) are also depicted in Figs.1-2. Fig.1 shows the dependence of the radii of marginally stable circular orbits on the rotation parameter a for values of $\epsilon = 0, 1$. In Fig.2 for various values of the black hole rotation parameter we illustrate the enlargement of the region of the marginal stability with the growth of the influence parameter of the magnetic field. In both cases we observe that even not very large value of the magnetic field produces an essential enlargement in the region of the marginally stable circular orbits towards the horizon of the black hole.

Next, substituting the value of the radius (41) into equations (23)-(24) we calculate the limiting value for the pertaining energy and angular momentum at a retrograde motion

$$E = 7.82\,\epsilon\,, \qquad L = -42.55\,\epsilon\,M. \qquad (43)$$

As $\epsilon \gg 1$, the retrograde motion of a charged particle at the last stable circular orbit around a maximally rotating black hole is of a relativistic motion

as in the case of a nonrotating black hole (see equation (39)), however, the rotation further enhances this effect. In the same way, using the radius r_2 in (42) we find the energy and the angular momentum

$$E = 0.41\,\epsilon\,, \qquad L = 3.82\,\epsilon\,M \qquad (44)$$

pertaining to the innermost direct, anti-Larmor, motion. It follows that the anti-Larmor motion of a charged particle at the radius r_2 may also become relativistic for $\epsilon \gg 1$, in contrast to that in the case of a nonrotating black hole, where the corresponding energy tends to zero (equation (40)).

In Tables 3.1-3.2 we also give the full list for the values of the energies and angular momenta pertaining to the marginally stable circular orbits. Comparing these quantities with those in Table 2 we see that in all cases the energy of the retrograde motion monotonically increases up to its relativistic values as the influence of the magnetic field, as well as the rotation of the black hole increase. However, the energy of the anti-Larmor motion around a Schwarzschild black hole systematically decreases as a test particle approaches the event horizon. The rotation of the black hole drastically changes the situation. The energies of direct motions increase monotonically as the hole's rotation parameter increases, excepting the nearest to the horzon region, where the effect of the expelling of the magnetic field becomes dominant (Bičák & Janiš 1985). Thus, we may conclude that for large enough values of the magnetic field strength, $\epsilon \gg 1$, both retrograde and direct motions of charged particles around a rotating black hole become relativistic along the innermost stable circular orbits. It is especially important to note the possibility of the relativistic stable direct orbits near the event horizon of a rotationg black hole in the presence of a strong magnetic field, unlike the case of a nonrotating black hole. This is due to the intimate connection between gravitation and electrodynamics that arises in the model considered here.

5   Conclusions

The main purpose of this paper was to study the combined effects of the rotation of a Kerr black hole and an external magnetic field on the marginally

stable circular motion of charged particles. We have presented general equations governing the energy, the angular momentum and the region of the marginal stability of circular orbits around a rotating black hole in a uniform magnetic field. The analytical results obtained for large enough values of the magnetic field strength and for a maximum value of the black hole angular momentum, as well as the numerical analysis performed in general case have shown that in all cases of the circular orbits the magnetic field essentially enlarges the region of their marginal stability towards the event horizon. As for the effect of the rotation of a black hole it has been found to be different for direct and retrograde motions: For retrograde motion the rotation opposes to the magnetic field in its stabilizing effect, though the latter always remains be dominant and for extreme values of the magnetic field and rotation parameter there exists a limiting value for the enlargement of the region of stability towards the event horizon. In the case of direct motion the presence of a rotation produces an additional shift of the stable orbits to the event horizon. However, for large enough values of the magnetic field, when the innermost stable motion occurs at a radius that lies very close to the event horizon, the magnetic field lines are expelled from the black hole as the hole's rotation parameter increases. Accordingly, the direct motion occurs in two different orbits; one of them is also expelled from the black hole, while the other one being affected less and less by the magnetic field approaches the event horizon as the rotation becomes maximum.

We have shown that the presence of a strong magnetic field around a rotating black holes provides the possibility of retrograde motion in the innermost stable relativistic orbits and the rotation of the black hole further enhances this effect. It is very important that, unlike the Schwarzschild case, the combined effects of a sufficiently large magnetic field and the rotation of a black hole also result in relativistic motions in the innermost stable direct orbits that lie very close to the event horizon. Thus, nearby the event horizon of a rotating black hole there may exist a source of synchrotron emission from relativistic charged particles moving in stable circular orbits. This, of course, may play an important role both in the searches for black holes and in making feasible the probes of the metric in the strong-gravity domain.

ACKNOWLEDGMENTS

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Table 1. Marginally stable circular orbits around a Kerr black hole in the absence of a magnetic field

$\epsilon = 0$	direct orbits			retrograde orbits		
a/M	r/M	E	L/M	r/M	E	L/M
0.0	6.0	0.94	3.46	6.0	0.94	-3.46
0.2	5.32	0.93	3.26	6.63	0.94	-3.64
0.4	4.61	0.92	3.03	7.25	0.95	-3.80
0.6	3.82	0.90	2.75	7.85	0.95	-3.96
0.8	2.90	0.87	2.37	8.43	0.95	-4.10
0.999	1.18	0.65	1.34	8.99	0.96	-4.24

Table 2. Marginally stable circular orbits around a nonrotating black hole in a uniform magnetic field

$a = 0$	anti-Larmor orbits			Larmor orbits		
ϵ	r/M	E	L/M	r/M	E	L/M
0.0	6.0	0.94	3.46	6.0	0.94	-3.46
0.2	3.98	0.78	3.51	4.63	1.57	-6.32
0.4	3.35	0.71	3.92	4.41	2.50	-10.35
0.6	3.05	0.65	4.34	4.35	3.52	-14.73
0.8	2.86	0.62	4.74	4.33	4.59	-19.27
1.0	2.74	0.58	5.18	4.32	5.67	-23.86
1.4	2.58	0.53	6.03	4.31	7.86	-33.11
2.0	2.43	0.49	7.19	4.30	11.16	-46.99
10.0	2.10	0.32	23.16	4.30	55.60	-234.42
100.0	2.01	0.29	203.01	4.30	556.00	-2343.99

Table 3.1. Marginally stable circular orbits around a Kerr black hole in a uniform magnetic field with $\epsilon = 1$.

$\epsilon = 1$	direct orbits			retrograde orbits		
a/M	r/M	E	L/M	r/M	E	L/M
0.0	2.74	0.58	5.18	4.32	5.67	-23.86
0.2	2.57	0.62	4.46	4.67	6.16	-27.55
0.4	2.38	0.65	3.76	4.99	6.61	-31.17
0.6	2.17	0.69	3.10	5.31	7.06	-35.02
0.8	1.89	0.71	2.38	5.61	7.49	-38.84
0.999	1.15	0.59	1.21	5.89	7.89	-42.58

Table 3.2. The influence of a sufficiently strong magnetic field on the marginally stable circular orbits around a Kerr black hole

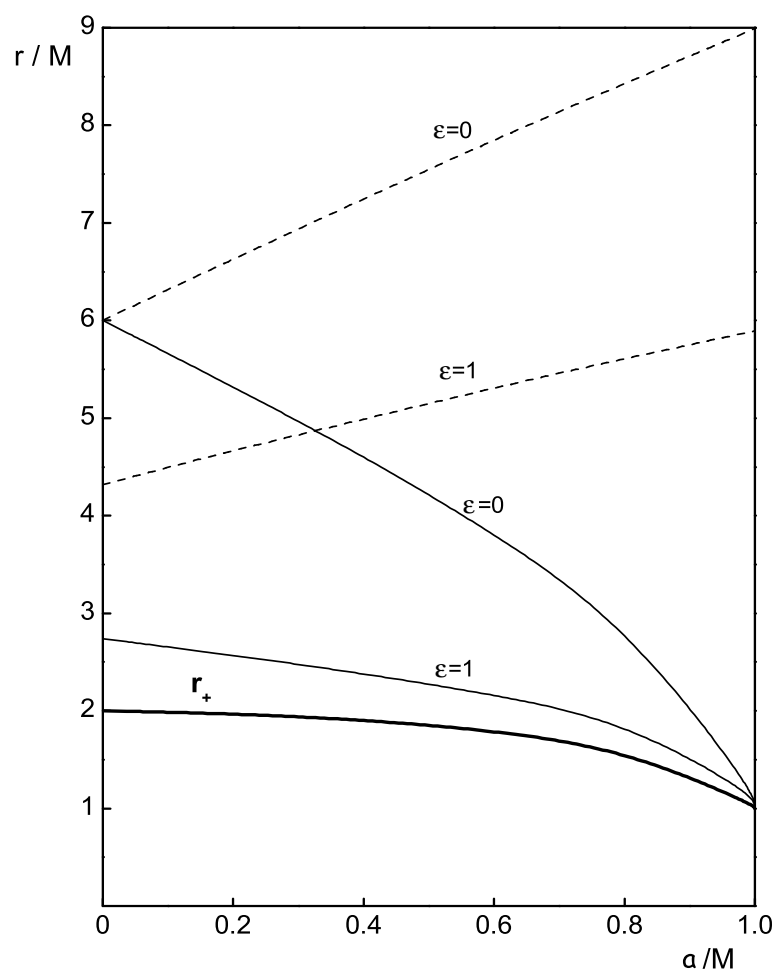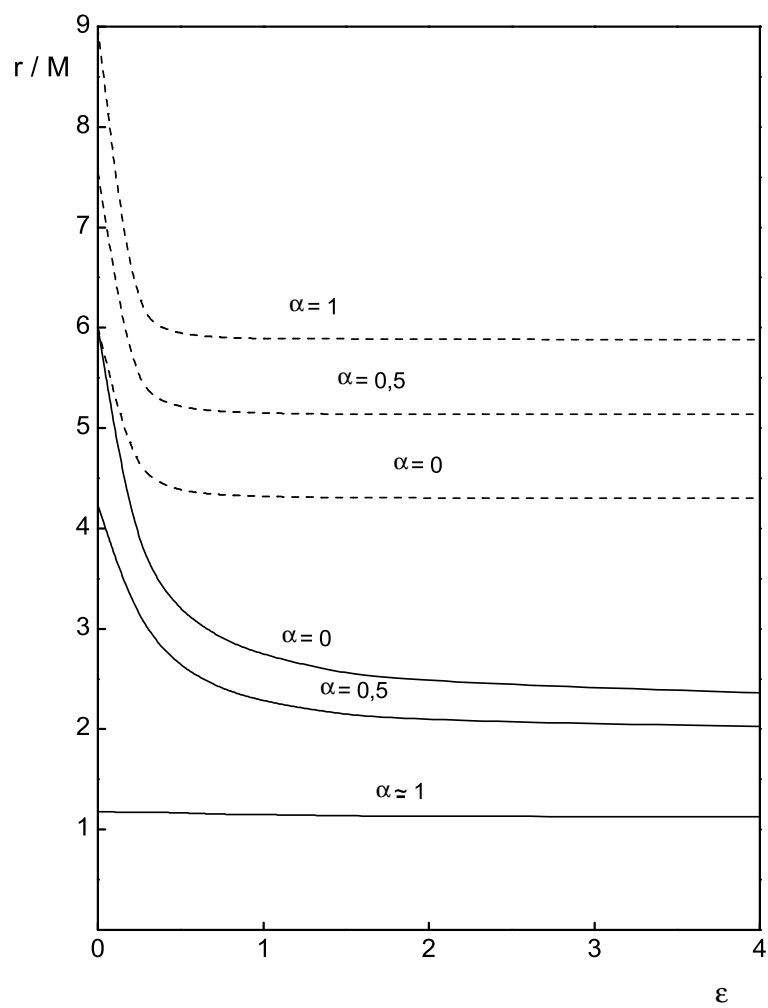| $\epsilon = 100$	direct		orbits				retrograde		orbits
a/M	r_1/M	E_1	L_1/M	r_2/M	E_2	L_2/M	r/M	E	L/M
0.0	2.01	0.29	203.01	2.01	0.29	203.01	4.30	556.00	-2343.99
0.2	1.98	9.97	195.55	2.05	9.49	219.31	4.65	605.31	-2714.25
0.4	1.93	19.74	185.63	2.13	18.46	246.97	4.98	652.19	-3090.56
0.6	1.84	29.09	168.84	2.23	26.44	288.74	5.29	696.60	-3468.64
0.8	1.68	37.34	141.35	2.35	33.32	343.57	5.59	739.89	-3857.50
0.999	1.13	36.15	73.75	2.47	39.55	405.10	5.88	782.03	-4255.20

Figure 1

Figure 2

Captions to figures;

Figure 1. The dependence of the radii of marginally stable circular orbits around a Kerr black hole on the rotation parameter a of the black hole for given $\epsilon = 0, 1$. The solid curves refer to the innermost direct orbits, while the dashed curves correspond to the innermost retrograde orbits, the position of the event horizon is shown by the bold curve r_+.

Figure 2. The dependence of the radii of marginally stable circular orbits around a Kerr black hole in a uniform magnetic field on the influence parameter of the magnetic field ϵ. The solid curves indicate the innermost direct orbits, while the dashed curves correspond to the innermost retrograde orbits $(\alpha = a/M = 0, 0.5, 1)$.